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Stability of the Cauchy Type Equations in the Class of Differentiable Functions

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Let X be a real normed space, Y a real Banach space, and let $C_n(X, Y)$ denote the space of n-times continuously differentiable functions $f: X \to Y$. We prove that the class C_n has the double difference property, that is if $\mathscr{C}f(x, y) := f(x+y) - f(x) - f(y)$ belongs to the space $C_n(X \times X, Y)$ then there exists an additive function $A: X \to Y$ such that $f - A \in C_n(X, Y)$. Similar result is also obtained for the Jensen equation. As an application we show that the Cauchy and Jensen equations are stable with respect to large class of seminorms defined by means of derivatives.

1. INTRODUCTION

In 1940, S. M. Ulam posed the following problem (cf. [11]).

We are given a group $(X, +)$ and a metric group $(Y, +, d)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f: X \to Y$ satisfies

$$d(f(x+y), f(x) + f(y)) < \delta \qquad \text{for all} \quad x, y \in X$$

then a homomorphism $a: X \to Y$ exists with

$$d(f(x), a(x)) < \varepsilon \qquad \text{for all} \quad x \in X?$$

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This question initiated the stability theory in the Hyers–Ulam. sense. The outline of this theory can be found in the survey papers [1, 2, 4].

Assume that Y is a normed space. For a function $f\colon X\to Y$ we put

$$\|f\|_{\sup} := \sup_{x\in X} \|f(x)\|.$$

Let $\mathscr{C}f$ denote the Cauchy difference of a function $f\colon X\to Y$, i.e, let

$$\mathscr{C}f(x, y) := f(x+y) - f(x) - f(y) \qquad \text{for} \quad x, y\in X.$$

Then the stability question can be reformulated as follows. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f\colon X\to Y$ satisfies

$$\|\mathscr{C}f\|_{\sup} < \delta$$

then an additive function $a\colon X\to Y$ exists with

$$\|f-a\|_{\sup} < \varepsilon?$$

It is natural to consider Ulam's question for different norms, not only for the supremum one. The stability problem with respect to $\mathscr{L}_p$ norm was considered in [9] and [6] and with respect to Lipschitz norms in [7].

Throughout the paper X and Y will denote a real normed space and a real Banach space, respectively. By $\mathbf{N}$ we denote the set of all nonnegative integers, and by $\mathbf{N}_+$ the set of all positive integers. Let $f\colon X\to Y$ be an n-times differentiable function. The nth derivative of f will by denoted by $D^n f$, and $D^0 f$ stands for f. By $C_n(X, Y)$ we denote the space of n-times continuously differentiable functions and by $BC_n(X, Y)$ the subspace of $C_n(X, Y)$ consisting of bounded functions. $C_\infty(X, Y)$ stands for the space of infinitely many times differentiable functions.

We assume that we are given a norm in $X\times X$ such that $\|(x_1, x_2)\|$ is a function of $\|x_1\|$ and $\|x_2\|$, and the following condition is satisfied:

$$\|(x, 0)\| = \|(0, x)\| = \|x\| \qquad \text{for} \quad x\in X.$$

For a function F defined in $X\times X$ its partial derivatives will be denoted by $\partial_1 F, \partial_2 F$. Let $i_1 : X\to X\times X$, $i_2 : X\to X\times X$ be injections defined by

$$i_1(x) := (x, 0) \qquad \text{for} \quad x\in X,$$

$$i_2(y) := (0, y) \qquad \text{for} \quad y\in X.$$

Let $L: X \times X \to Y$ be a bounded linear mapping. It follows directly from the assumed conditions on the norm in $X \times X$ that

$$\|L \circ i_1\| \leqslant \|L\| \|i_1\| = \|L\|,$$
$$\|L \circ i_2\| \leqslant \|L\| \|i_2\| = \|L\|.$$

Therefore if $F: X \times X \to Y$ is n-times differentiable $(n \geqslant 1)$ then

$$\begin{aligned} \|\partial_1 F(x, y)\| &= \|DF(x, y) \circ i_1\| \leqslant \|DF(x, y)\|, \\ \|\partial_2 F(x, y)\| &= \|DF(x, y) \circ i_2\| \leqslant \|DF(x, y)\|, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \|\partial_1^{i-1} \partial_2 F(x, y)\| &\leqslant \|D^i F(x, y)\|, \\ \|\partial_1 \partial_2^{i-1} F(x, y)\| &\leqslant \|D^i F(x, y)\| \end{aligned} \tag{1.2}$$

for $i = 1, ..., n$.

Let $f: X \to Y$ be any function. We define the Jensen difference of f by

$$\mathscr{I}f(x, y) := f\left(\frac{x+y}{2}\right) - \frac{f(x) + f(y)}{2},$$

for $(x, y) \in X \times X$.

Condition $\mathscr{C}f = 0$ $(\mathscr{I}f = 0)$ mean, that f satisfies the Cauchy (Jensen) equation.

Let $n \in N$, and let $f: X \to Y$ be n-times differentiable. Then $\mathscr{C}f$ and $\mathscr{I}f$ are n-times differentiable, and by (1.1) we have

$$\begin{aligned} \|Df(x+y) - Df(y)\| &\leqslant \|D(\mathscr{C}f)(x, y)\|, \\ \left\| Df\left(\frac{x+y}{2}\right) - Df(y) \right\| &\leqslant 2 \, \|D(Jf)(x, y)\| \end{aligned} \tag{1.3}$$

for $x, y \in X$.

Moreover, for $n \geqslant 2$, we obtain from (1.2)

$$\begin{aligned} \|D^i f(x+y)\| &\leqslant \|D^i(\mathscr{C}f)(x, y)\|, \\ \left\| D^i f\left(\frac{x+y}{2}\right) \right\| &\leqslant 2^i \, \|D^i(Jf)(x, y)\| \end{aligned} \tag{1.4}$$

for $x, y \in X$, $i = 2, ..., n$.

To avoid distinguishing some cases and to shorten some considerations we will use the following convention. If $m, n \in N$, $m > n$ then by $\sum_{i=m}^{n} a_i$ we mean zero.

2. DOUBLE DIFFERENCE PROPERTY

We will prove that the class $C_n(X, Y)$ has the so called "double difference property" (cf. [5]), i.e., if $f: X \to Y$ is such a function that $\mathscr{C}f \in C_n(X \times X, Y)$, then there exists an additive function $A: X \to Y$ such that $f - A \in C_n(X, Y)$. A similar property for the Jensen difference will be proved, too. These results will enable us to solve the stability problem of the Cauchy and Jensen equations, with respect to certain class of seminorms in $C_n(X, Y)$.

THEOREM 2.1. *Let $n \in \mathbf{N}_+ \cup \{\infty\}$, and let $f: X \to Y$ be such a function that $\mathscr{C}f \in C_n(X \times X, Y)$. Then there exists a unique additive function $A_0: X \to Y$ such that $f - A_0 \in C_n(X, Y)$ and $D(f - A_0)(0) = 0$. Moreover, then*

$$\|D^k(f - A_0)(0)\| \leqslant \|D^k \mathscr{C}f(0)\| \qquad \text{for} \quad k \in \mathbf{N}, \qquad k \leqslant n, \tag{2.1}$$

$$\|D^k(f - A_0)\|_{\sup} \leqslant \|D^k \mathscr{C}f\|_{\sup} \qquad \text{for} \quad k \in \mathbf{N} \backslash \{0\}, \quad k \leqslant n. \tag{2.2}$$

Proof. We show the first part of the theorem. Let $f_1 = f - f(0)$. Then $\mathscr{C}f_1 = \mathscr{C}f + f(0) \in \mathscr{C}_n(X \times X, Y)$ and $\mathscr{C}f_1(0, 0) = 0$. Let $x, y \in X$ be arbitrarily fixed. We consider the function

$$\varphi(t) := \mathscr{C}f_1(tx, ty) \qquad \text{for} \quad t \in \mathbf{R}.$$

Then we obtain

$$\mathscr{C}f_1(x, y) = \varphi(1) - \varphi(0) = \int_0^1 D\varphi(t)\, dt = \int_0^1 D(\mathscr{C}f_1)(tx, ty)(x, y)\, dt$$

$$= \int_0^1 \partial_2(\mathscr{C}f_1)(ty, tx)(x)\, dt + \int_0^1 \partial_2(\mathscr{C}f_1)(tx, ty)(y)\, dt.$$

Thus

$$(\mathscr{C}f_1)(x, y) = \int_0^1 \partial_2(\mathscr{C}f_1)(ty, tx)(x)\, dt + \int_0^1 \partial_2(\mathscr{C}f_1)(tx, ty)(y)\, dt \qquad \text{for} \quad x, y \in X. \tag{2.3}$$

Notice that

$$\mathscr{C}f_1(x + y, z) + \mathscr{C}f_1(x, y) = \mathscr{C}f_1(x, y + z) + \mathscr{C}f_1(y, z) \qquad \text{for} \quad x, y, z \in X.$$

Differentiating both sides of this equality with respect to z at the point $z=0$, we obtain

$$\partial_2(\mathscr{C}f_1)(x+y,0)=\partial_2(\mathscr{C}f_1)(x,y)+\partial_2(\mathscr{C}f_1)(y,0) \qquad \text{for} \quad x,y\in X. \tag{2.4}$$

We define $A_0: X\to Y$ by the formula

$$A_0(x):=f_1(x)-\int_0^1 \partial_2(\mathscr{C}f_1)(tx,0)(x)\,dt \qquad \text{for} \quad x\in X.$$

We show that A_0 is additive. Making use of (2.3) and (2.4) we obtain for $x, y\in X$

$$\begin{aligned}
&A_0(x+y)-A_0(x)-A_0(y)\\
&\quad=\mathscr{C}f_1(x,y)-\int_0^1 \partial_2(\mathscr{C}f_1)(t(x+y),0)(x+y)\,dt\\
&\qquad+\int_0^1 \partial_2(\mathscr{C}f_1)(tx,0)(x)\,dt+\int_0^1 \partial_2(\mathscr{C}f_1)(ty,0)(y)\,dt\\
&\quad=\int_0^1 \partial_2(\mathscr{C}f_1)(ty,tx)(x)\,dt+\int_0^1 \partial_2(\mathscr{C}f_1)(tx,ty)(y)\,dt\\
&\qquad-\int_0^1 \partial_2(\mathscr{C}f_1)(tx+ty,0)(x)\,dt-\int_0^1 \partial_2(\mathscr{C}f_1)(tx+ty,0)(y)\,dt\\
&\qquad+\int_0^1 \partial_2(\mathscr{C}f_1)(tx,0)(x)\,dt+\int_0^1 \partial_2(\mathscr{C}f_1)(ty,0)(y)\,dt\\
&\quad=\int_0^1 (\partial_2(\mathscr{C}f_1)(ty,tx)+\partial_2(\mathscr{C}f_1)(tx,0)-\partial_2(\mathscr{C}f_1)(ty+tx,0))(x)\,dt\\
&\qquad+\int_0^1 (\partial_2(\mathscr{C}f_1)(tx,ty)+\partial_2(\mathscr{C}f_1)(ty,0)-\partial_2(\mathscr{C}f_1)(tx+ty,0))(y)\,dt=0.
\end{aligned}$$

It means that A_0 is additive.

Consider arbitrary $x, h\in X$. We have

$$\mathscr{C}f_1(x,h)=\mathscr{C}f_1(x,h)-\mathscr{C}f(x,0)=\int_0^1 \partial_2(\mathscr{C}f_1)(x,th)(h)\,dt.$$

Further by (2.4) we obtain

$$\int_0^1 \partial_2(\mathscr{C}f_1)(x+th,0)(h)\,dt = \int_0^1 \partial_2(\mathscr{C}f_1)(x,th)(h)\,dt + \int_0^1 \partial_2(\mathscr{C}f_1)(th,0)(h)\,dt$$

$$= \mathscr{C}f_1(x,h) + \int_0^1 \partial_2(\mathscr{C}f_1)(th,0)(h)\,dt.$$

Thus we have

$$\int_0^1 \partial_2(\mathscr{C}f_1)(th,0)(h)\,dt = -\mathscr{C}f_1(x,h) + \int_0^1 \partial_2(\mathscr{C}f_1)(x+th,0)(h)\,dt \tag{2.5}$$

for $x, h \in X$.

Now we prove that $f - A_0$ is differentiable. Consider arbitrary $x, h \in X$. Making use of the additivity of A_0 and (2.5) we obtain

$$\|f_1(x+h) - A_0(x+h) - (f_1(x) - A_0(x)) - \partial_2(\mathscr{C}f_1)(x,0)(h)\|$$

$$= \left\| \mathscr{C}f_1(x,h) + \int_0^1 \partial_2(\mathscr{C}f_1)(th,0)(h)\,dt - \partial_2(\mathscr{C}f_1)(x,0)(h) \right\|$$

$$= \left\| \int_0^1 (\partial_2(\mathscr{C}f_1)(x+th,0) - \partial_2(\mathscr{C}f_1)(x,0)(h))\,dt \right\|$$

$$\leqslant \|h\| \sup_{t\in[0,1]} \|\partial_2(\mathscr{C}f_1)(x+th,0) - \partial_2(\mathscr{C}f_1)(x,0)\|.$$

Since $\partial_2(\mathscr{C}f_1) = \partial_2(\mathscr{C}f)$ is continuous, $\|\partial_2(\mathscr{C}f_1)(x+th,0) - \partial_2(\mathscr{C}f_1)(x,0)\|$ is small for small h. Hence the function $f - A_0 = f_1 - A_0 + f(0)$ is differentiable at x and

$$D(f - A_0)(x) = \partial_2(\mathscr{C}f_1)(x,0) = \partial_2(\mathscr{C}f)(x,0).$$

But $\partial_2(\mathscr{C}f) \in C_{n-1}(X \times X, Y)$, and hence $\partial_2(\mathscr{C}f)(x,0) \in C_{n-1}(X, Y)$. Therefore $D(f - A_0) \in C^{n-1}(X, Y)$, i.e., $f - A_0 \in C^n(X, Y)$. Moreover we have

$$D(f - A_0)(0) = \partial_2(\mathscr{C}f)(0,0) = 0.$$

To prove the uniqueness of A_0 consider additive functions $A_1, A_2 : X \to Y$ such that $f - A_1, f - A_2 \in C_n(X, Y)$ and $D(f - A_1)(0) = D(f - A_2)(0) = 0$. Then $A_1 - A_2$ is additive, $A_1 - A_2 \in C_n(X, Y)$ and $D(A_1 - A_2)(0) = 0$. It yields that $A_1 = A_2$.

We claim that

$$\|D^k(f-A_0)(x)\| \leqslant \|D^k(\mathscr{C}f)(x,0)\| \qquad \text{for} \quad x \in X, \quad k \leqslant n, \quad k \in \mathbf{N}\setminus\{0\}. \tag{2.6}$$

Making use of (1.3) and the fact that $D(f-A_0)(0)=0$, we obtain for $x \in X$

$$\|D(f-A_0)(x)\| = \|D(f-A_0)(x) - D(f-A_0)(0)\| \leqslant \|D(\mathscr{C}f)(x,0)\|,$$

which proves (2.6) for $k=1$. For $2 \leqslant k \leqslant n$, $k \in \mathbf{N}$ (2.6) follows directly from (1.4).

The case $k=0$ in (2.1) is trivial. From (2.6) we obtain (2.1) for $k \geqslant 1$ and (2.2). ∎

Theorem 2.1 states, in particular, that the class of infinitely many times differentiable functions has the double difference property. It is natural to ask whether the class of analytic functions has this property. We show that the answer is positive.

COROLLARY 2.1. *Let $f: X \to Y$ be a function such that $\mathscr{C}f$ is analytic. Then there exists a unique additive function $A_0: X \to Y$ such that $f - A_0$ is analytic and $D(f-A_0)(0)=0$.*

Proof. By Theorem 2.1 there exists a unique additive function A_0: $X \to Y$ such that $f_1 := f - A_0 \in C_\infty(X, Y)$ and $Df_1(0)=0$. Then obviously $\mathscr{C}f_1 = \mathscr{C}f$, and hence $\mathscr{C}f_1$ is analytic. Making use of the equality

$$\partial_2 \mathscr{C}f_1(x,0) = Df_1(x) - Df_1(0) \qquad \text{for} \quad x \in X,$$

we obtain that Df_1 is analytic, and consequently that f_1 is analytic. ∎

In further considerations the following lemma will play an essential role.

LEMMA 2.1. *Let $A: X \to Y$ be an additive function, and let $f: X \to Y$ be a differentiable function such that $f - A$ is bounded. Let $\varepsilon \in \mathbf{R}_+ \cup \{\infty\}$ be such that*

$$\sup_{(x,y) \in X \times X} \|Df(x) - Df(y)\| \leqslant \varepsilon. \tag{2.7}$$

Then A is linear continuous and

$$\sup_{x \in X} \|Df(x) - A\| \leqslant \varepsilon.$$

Proof. Since $f-A$ is bounded,

$$A(x)=\lim_{n\to\infty}\frac{f(nx)}{n} \qquad \text{for} \quad x\in X. \tag{2.8}$$

By the Theorems 1 and 2 of D. H. Hyers (cf. [3]) A is linear continuous. We fix arbitrary $y\in X$. Applying the Mean Value Theorem for the function $f(x)-Df(y)(x)$ and (2.7) we obtain

$$\|f(x)-Df(y)(x)-f(0)\|\leqslant(\sup_{x\in X}\|Df(x)-Df(y)\|)\ \|x\|\leqslant\varepsilon\ \|x\|.$$

Replacing in this inequality x by nx and dividing both sides by n we get

$$\left\|\frac{f(nx)}{n}-Df(y)(x)-\frac{f(0)}{n}\right\|\leqslant\varepsilon\ \|x\| \qquad \text{for} \quad x\in X.$$

Letting $n\to\infty$ and applying (2.8) we conclude that

$$\|A(x)-Df(y)(x)\|\leqslant\varepsilon\ \|x\| \qquad \text{for} \quad x\in X,$$

which means that $\|A-Df(y)\|\leqslant\varepsilon$. Since y was arbitrary, we have

$$\sup_{y\in X}\|A-Df(y)\|\leqslant\varepsilon. \quad \blacksquare$$

THEOREM 2.2. *Let $n\in\mathbf{N}_+\cup\{\infty\}$, and let $f\colon X\to Y$ be such a function that $\mathscr{C}f\in BC_n(X\times X, Y)$. Then there exists a unique additive function A_∞: $X\to Y$ such that $f-A_\infty\in BC_n(X, Y)$. Moreover, then*

$$\|D^k(f-A_\infty)\|_{\sup}\leqslant\|D^k\mathscr{C}f\|_{\sup} \qquad \textit{for} \quad k\in\mathbf{N}, \qquad k\leqslant n, \tag{2.9}$$

$$\|D^k(f-A_\infty)(0)\|\leqslant\|D^k\mathscr{C}f(0)\| \qquad \textit{for} \quad k\in\mathbf{N}\setminus\{1\}, \quad k\leqslant n. \tag{2.10}$$

Proof. In virtue of Theorem 2.1 there exists an additive function A_0: $X\to Y$ such that $f_1:=f-A_0\in C_n(X, Y)$. Then $\mathscr{C}f_1=\mathscr{C}f\in BC_n(X\times X, Y)$. It means, in particular, that $\mathscr{C}f_1$ is bounded. By the Hyers Theorem there exists an additive function $A_1: X\to Y$ such that

$$\sup_{x\in X}\|f_1(x)-A_1(x)\leqslant\sup_{(x, y)\in X\times X}\|\mathscr{C}f_1(x, y)\|. \tag{2.11}$$

We put

$$A_\infty(x):=A_0(x)+A_1(x) \qquad \text{for} \quad x\in X.$$

Clearly A_∞ is additive and $f - A_\infty = f_1 - A_1$. From (1.3) we obtain

$$\sup_{(x, y) \in X \times X} \|Df_1(x) - Df_1(y) \leqslant \sup_{(x, y) \in X \times X} \|D(\mathscr{C}f_1)(x, y)\|. \tag{2.12}$$

Conditions (2.11) and (2.12) mean that the functions f_1 and A_1 satisfy the assumptions of Lemma 2.1 with $\varepsilon = \sup_{(x, y) \in X \times X} \|D(\mathscr{C}f_1)(x, y)\|$. Hence A_1 is continuous linear and

$$\sup_{x \in X} \|D(f - A_\infty)(x)\| = \sup_{x \in X} \|Df_1(x) - A_1\|$$

$$\leqslant \sup_{(x, y) \in X \times X} \|D(\mathscr{C}f)(x, y)\|. \tag{2.13}$$

Making use of (2.11), (2.13), and (1.4) we obtain (2.9). For $k = 0$ (2.10) is obvious and for $k \geqslant 2$ it is a trivial consequence of (1.4). ∎

PROPOSITION 2.1. *The functions A_0 and A_∞ occuring in Theorems* 2.1 *and* 2.2 *can be defined by the formulae*

$$A_0(x) =: \lim_{n \to \infty} \frac{f(x/n) - f(0)}{1/n} \qquad \textit{for} \quad x \in X,$$

$$A_\infty(x) = \lim_{n \to \infty} \frac{f(nx)}{n} \qquad \textit{for} \quad x \in X.$$

Proof. Since $D(f - A_0)(0) = 0$, we have for $x \in X$, $x \neq 0$

$$\lim_{n \to \infty} \frac{\|f(x/n) - A_0(x/n) - f(0)\|}{\|x/n\|} = 0,$$

i.e.,

$$\lim_{n \to \infty} \left\| A_0(x) - n\left(f\left(\frac{x}{n}\right) - f(0)\right) \right\| \cdot \frac{1}{\|x\|} = 0,$$

which yields the first formula for $x \in X$, $x \neq 0$. It obviously also holds for $x = 0$.

The formula for A_∞ is a trivial consequence of the fact that the function $f - A_\infty$ is bounded. ∎

Comparing the formulae for A_0 and A_∞ one can easily notice that these functions are usually different. The following example shows it explicitly.

EXAMPLE 2.1. Let $f: X \to Y$ be a bounded differentiable function. As f is bounded, we obtain that $A_\infty = 0$. Since $A_0 = Df(0)$, $A_0 = A_\infty$ if and only if $Df(0) = 0$.

3. JENSEN DIFFERENCE PROPERTY

Now we are going to prove analogues of Section 2 for the Jensen difference.

THEOREM 3.1. *Let $n \in \mathbf{N}_+ \cup \{\infty\}$, and let $f: X \to Y$ be such a function that $\mathscr{J}f \in C_n(X \times X, Y)$. Then there exists a unique Jensen function F_0: $X \to Y$ such that $f - F_0 \in C_n(X, Y)$ and $f(0) = F_0(0)$, $D(f - F_0)(0) = 0$. Moreover, then*

$$\|D^k(f - F_0)(0)\| \leqslant 2^k \|D^k \mathscr{J}f(0)\| \quad \textit{for} \quad k \in \mathbf{N}, \quad k \leqslant n, \tag{3.1}$$

$$\|D^k(f - F_0)\|_{\sup} \leqslant 2^k \|D^k \mathscr{J}f\|_{\sup} \quad \textit{for} \quad k \in \mathbf{N} \setminus \{0\}, \quad k \leqslant n. \tag{3.2}$$

Proof. We have for $x, y \in X$

$$\mathscr{C}f(x, y) = f(x + y) - f(x) - f(y) = \mathscr{J}f(2x, 2y) - \mathscr{J}f(2x, 0) - \mathscr{J}f(0, 2y).$$

Hence $\mathscr{C}f \in \mathscr{C}_n(X \times X, Y)$. By Theorem 2.1 there exists a unique additive function $A_0: X \to Y$ such that $f - A_0 \in \mathscr{C}_n(X, Y)$ and $D(f - A_0)(0) = 0$. We put $F_0 = A_0 + f(0)$. Since any Jensen function is a sum of an additive function and a constant it is clear that the conditions $F_0(0) = f(0)$, $f - F_0 \in C_n(X, Y)$ and $D(f - F_0)(0) = 0$ determine F_0 uniquely. Applying these results, (1.3) and (1.4) we obtain

$$\|D^k(f - F_0)(x)\| \leqslant 2^k \|D^k(\mathscr{J}f)(2x, 0)\| \quad \text{for} \quad x \in X, \quad k \in \mathbf{N}, \quad k \leqslant n.$$

Thus we get (3.1) for $k \geqslant 1$ and (3.2). (3.1) for $k = 0$ is obvious as $(f - F_0)(0) = 0$. ∎

The stability question of the Jensen equation with respect to the supremum norm has already been solved (cf. [3], [9], [1]). It was done by reducing the problem to the Cauchy case. The estimation of $f - F$ obtained in this way, namely $\|f - F\|_{\sup} \leqslant 4 \|\mathscr{J}f\|_{\sup}$ is not sharp. We obtain a better one (we are indebted to the referee who shortened our original proof of this result).

PROPOSITION 3.1. *Let $(S, +)$ be a uniquely 2-divisible commutative semigroup with zero. If a function $f: S \to Y$ satisfies the inequality*

$$\|\mathscr{J}f\|_{\sup} < \infty,$$

then there exists a unique Jensen function $F: S \to Y$ *such that* $F(0) = f(0)$ *and*

$$\|f - F\|_{\sup} \leqslant 2 \, \|\mathscr{I}f\|_{\sup}.$$

Proof. We know that there is a Jensen function $F: S \to Y$ such that $F(0) = f(0)$ and $g = f - F$ is bounded. We have to show that

$$\|g\|_{\sup} \leqslant 2 \, \|\mathscr{I}f\|_{\sup}.$$

We may assume that $\|\mathscr{I}f\|_{\sup} = 1$. Since $g(0) = 0$ and $\mathscr{I}g = \mathscr{I}f$, we have

$$\|g(x) - \tfrac{1}{2} g(2x)\| \leqslant \|\mathscr{I}g\|_{\sup} = 1$$

for every $x \in S$. Therefore

$$\|g(x)\| \leqslant 1 + \tfrac{1}{2} \, \|g\|_{\sup}.$$

Since this is true for every $x \in S$, it follows that $\|g\|_{\sup} \leqslant 1 + \frac{1}{2} \, \|g\|_{\sup}$ and $\|g\|_{\sup} \leqslant 2$.

The uniqueness of F is trivial. ∎

THEOREM 3.2. *Let* $n \in \mathbf{N}_+ \cup \{\infty\}$, *and let* $f: X \to Y$ *be such a function that* $\mathscr{I}f \in BC_n(X \times X, Y)$. *Then there exists a unique Jensen function* $F_\infty : X \to Y$ *such that* $f - F_\infty \in BC_n(X, Y)$ *and* $F_\infty(0) = f(0)$. *Moreover,*

$$\|D^k(f - F_\infty)\|_{\sup} \leqslant \max\{2, 2^k\} \, \|D^k \mathscr{I}f\|_{\sup} \quad \text{for} \quad k \leqslant n, \quad k \in \mathbf{N}, \tag{3.3}$$

$$\|D^k(f - F_\infty)(0)\| \leqslant 2^k \, \|D^k \mathscr{I}f(0)\| \quad \text{for} \quad k \leqslant n, \quad k \in \mathbf{N} \backslash \{1\}. \tag{3.4}$$

Proof. In virtue of Theorem 3.1 there exists a Jensen function $F_0 : X \to Y$ such that $f_1 := f - F_0 \in C_n(X, Y)$ and $f_1(0) = 0$. Then $\mathscr{I}f_1 = \mathscr{I}f \in BC_n(X \times X, Y)$. It implies that $\mathscr{I}f_1$ is bounded. By Proposition 3.1 there exists a Jensen function $F_1 : X \to Y$ such that $F_1(0) = f_1(0) = 0$ and

$$\sup_{x \in X} \|f_1(x) - F_1(x)\| \leqslant 2 \sup_{(x, y) \in X \times X} \|\mathscr{I}f_1(x, y)\|. \tag{3.5}$$

We put

$$F_\infty(x) := F_0(x) + F_1(X) \qquad \text{for} \quad x \in X.$$

Clearly F_∞ is Jensen and $f-F_\infty=f_1-F_1$. From (1.3) we obtain

$$\sup_{(x,y)\in X\times X}\|Df_1(x)-Df_1(y)\|\leqslant 2\sup_{(x,y)\in X\times X}\|D(\mathscr{I}f_1)(x,y)\|. \tag{3.6}$$

As F_1 is Jensen and $F_1(0)=0$ we obtain that F_1 is additive. Conditions (3.5) and (3.6) imply that the functions f_1 and F_1 satisfy the assumptions of Lemma 2.1 with $\varepsilon=2\sup_{(x,y)\in X\times X}\|\mathscr{I}f\|$. Hence F_1 is continuous linear and

$$\sup_{x\in X}\|D(f-F_\infty)(x)\|=\sup_{x\in X}\|Df_1(x)-F_1\|\leqslant 2\sup_{(x,y)\in X\times X}\|D(\mathscr{I}f)(x,y)\|. \tag{3.7}$$

Making use of (3.5), (3.7), and (1.4) we obtain (3.3). For $k=0$ (3.4) is trivial and for $k\geqslant 2$ it is a direct consequence of (1.4).

The uniqueness of F follows from Proposition 3.1. ▮

4. STABILITY

In subspaces of $C_n(X,Y)$ one can consider different norms defined in terms of $\|D^if(0)\|$, $\|D^if\|_{\sup}$ for $i\leqslant n$. For example the following norms

$$\|f\|:=\sum_{i=0}^{n-1}\|D^if(0)\|+\|D^nf\|_{\sup}, \tag{4.1}$$

$$\|f\|:=\max_{i=0,\dots,n}\|D^if\|_{\sup} \tag{4.2}$$

are used very often. Obviously several other natural norms can be introduced. Our aim is not to restrict to any particular norm, and therefore we will formulate our results in a possibly general setting.

Let $n\in\mathbf{N}\cup\{\infty\}$ be fixed. In the set $[0,\infty]^{2n}$ we introduce the following order

$$(x_1,x_2,\dots)\leqslant(y_1,y_2,\dots)$$

iff $x_i\leqslant y_i$ for $i\in\mathbf{N}$, $i\leqslant 2n$.

Let $p\colon[0,\infty]^{2n}\to[0,\infty]$ be any function satisfying the following conditions

(i) $p(x+y) \leqslant p(x)+p(y)$ for $x, y \in [0, \infty]^{2n}$,

(ii) $p(\alpha x)=\alpha p(x)$ for $x \in [0, \infty]^{2n}$, $\alpha \in [0, \infty)(0 \cdot \infty = 0)$,

(iii) $x \leqslant y \Rightarrow p(x) \leqslant p(y)$ for $x, y \in [0, \infty]^{2n}$.

From (ii) we obtain that $p(0)=0$.

We define the mapping $\Phi\colon C_n(X, Y) \to [0, \infty]^{2n}$ by the formula

$$\Phi(f) := (\|f(0)\|, \|f\|_{\sup}, \|Df(0)\|, \|Df\|_{\sup}, ...)$$

and put

$$S_p(X, Y) := \{f \in C_n(X, Y)\colon p(\Phi)(f)) < \infty\}.$$

Since $p(0)=0$, S_p contains at least the zero function. It is easy to notice that S_p is a linear space and that $p \circ \Phi \mid S_p$ is a seminorm. We will denote this seminorm by $\|\cdot\|_p$. The same notations we will apply for the space $C_n(X \times X, Y)$.

THEOREM 4.1. *Let $f\colon X \to Y$ be such a function that $\mathscr{C}f \in S_p(X \times X, Y)$. We additionally assume that the function p does not depend on the second or third variable.*

Then there exists an additive function $A\colon X \to Y$ such that $f - A \in S_p(X, Y)$ and

$$\|f - A\|_p \leqslant \|\mathscr{C}f\|_p.$$

Proof. Assume that $\mathscr{C}f \in S_p(X \times X, Y)$. Suppose that p does not depend on the second variable. Then $p(0, \infty, 0, ...) = p(0, 0, 0, ...) = 0$. By Theorem 2.1 there exists an additive function $A_0\colon X \to Y$ satisfying conditions (2.1) and (2.2). Then $\Phi(f - A_0) \leqslant \Phi(\mathscr{C}f) + (0, \infty, 0, 0, ...)$ and hence

$$p(\Phi(f - A_0)) \leqslant p(\phi(\mathscr{C}f) + (0, \infty, 0, ...)) \leqslant p(\Phi)(\mathscr{C}f_0)) + 0,$$

i.e.,

$$\|f - A_0\|_p \leqslant \|\mathscr{C}f\|_p.$$

Suppose now that p does not depend on the third variable. If $\mathscr{C}f \in BC_n(X \times X, Y)$ then by Theorem 2.2 there exists an additive function A_∞ such that conditions (2.9) and (2.10) are satisfied. Hence $\Phi(f - A_\infty) \leqslant \Phi(\mathscr{C}f) + (0, 0, \infty, 0, ...)$ and consequently

$$\|f - A_\infty\|_p \leqslant \|\mathscr{C}f\|_p.$$

If $\mathscr{C}f$ is unbounded then $\|\mathscr{C}f\|_{\sup}=\infty$. By Theorem 2.1 we can find an additive function such that the conditions (2.1) and (2.2) hold. Then $\Phi(f-A_0)\leqslant\Phi(\mathscr{C}f)$ and hence

$$\|f-A_0\|_p\leqslant\|\mathscr{C}f\|_p.$$

One can easily notice that if we define for $n\in\mathbf{N}_+$

$$p(x_1, x_2, ..., x_{2n}):=\sum_{i=0}^{n-1} x_{2i+1}+x_{2n}, \tag{4.3}$$

or

$$p(x_1, x_2, ..., x_{2n}):=\max_{i=1, ..., n}\{x_{2i}\}, \tag{4.4}$$

then we would obtain stability of the Cauchy equation in the norms defined by the formulae (4.1) or (4.2).

The following example shows that the estimation of $f-A$ obtained in Theorem 4.1 for $n=2$ and the norm defined by the formula (4.4) is sharp.

EXAMPLE 4.1. Let $n=2$ and let p be defined as in (4.4). Let the norm in $X\times X$ be defined by

$$\|(x_1, x_2)\|=|x_1|+|x_2| \qquad \text{for} \quad x_1, x_2\in\mathbf{R},$$

and let

$$f(x)=\arctan x \qquad \text{for} \quad x\in\mathbf{R}.$$

One can check easily that for a fixed $y\in\mathbf{R}$

$$\sup_{x\in\mathbf{R}}|\mathscr{C}f(x, y)|=|\arctan|.$$

Hence

$$\sup_{x\in\mathbf{R}}|\mathscr{C}f(x, y)|=\frac{\pi}{2}.$$

We will calculate $\sup_{(x, y)\in\mathbf{R}\times\mathbf{R}}\|D(\mathscr{C}f)(x, y)\|$. Fix arbitrarily $x, y\in\mathbf{R}$. Then we obtain

$$\|D(\mathscr{C}f)(x, y)\| = \sup_{\|(h_1, h_2)\|=1} \left| \left(\frac{1}{1+(x+y)^2} - \frac{1}{1+x^2} \right) h_1 \right.$$
$$\left. + \left(\frac{1}{1+(x+y)^2} - \frac{1}{1+y^2} \right) h-2 \right|$$
$$\geqslant \max \left\{ \left| \frac{1}{1+(x+y)^2} - \frac{1}{1+x^2} \right|, \left| \frac{1}{1+(x+y)^2} - \frac{1}{1+y^2} \right| \right\}.$$

On the other hand, we have

$$\|D(\mathscr{C}f)(x, y)\| \leqslant \sup_{\|(h_1, h_2)\|=1} \left(\left| \frac{1}{1+(x+y)^2} - \frac{1}{1+x^2} \right| |h_1| \right.$$
$$\left. + \left| \frac{1}{1+(x+y)^2} - \frac{1}{1+y^2} \right| |h_2| \right)$$
$$\leqslant \max \left\{ \left| \frac{1}{1+(x+y)^2} - \frac{1}{1+x^2} \right|, \left| \frac{1}{1+(x+y)^2} - \frac{1}{1+y^2} \right| \right\}.$$

Thus we have

$$\|D(\mathscr{C}f)(x, y)\| = \max \left\{ \left| \frac{1}{1+(x+y)^2} - \frac{1}{1+x^2} \right|, \left| \frac{1}{1+(x+y)^2} - \frac{1}{1+y^2} \right| \right\}$$

for $x, y \in \mathbf{R}$.

Hence we obtain

$$\sup_{(h_1, h_2) \in \mathbf{R} \times \mathbf{R}} \|D(\mathscr{C}f)(x, y)\| = 1,$$

and consequently

$$\|\mathscr{C}f\|_p = \max \left\{ \frac{\pi}{2}, 1 \right\} = \frac{\pi}{2}.$$

Since f is bounded, the unique additive function which approximates f is $A \equiv 0$. Then

$$\|f - A\|_p = \max\{ \sup_{x \in \mathbf{R}} |f(x)|, \sup_{x \in \mathbf{R}} |Df(x)| \} = \max \left\{ \frac{\pi}{2}, 1 \right\} = \frac{\pi}{2}.$$

Making use of Theorems 3.1, 3.2 by the similair reasoning as in Theorem 4.1, one can get the following stability result on the Jensen equation.

THEOREM 4.2. *Let $n \in \mathbf{N}_+$, and let $f\colon X \to Y$ be such a function that $\mathscr{I}f \in S_p(X \times X, Y)$. We additionally assume that p does not depend on the second or third variable.*

Then there exists a Jensen function $F\colon X \to Y$ such that $f - F \in S_p(X, Y)$ and

$$\|f - F\|_p \leqslant 2^n \, \|\mathscr{I}f\|_p.$$

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